WEITZENBÖCK DERIVATIONS OF FREE METABELIAN LIE ALGEBRAS

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ABSTRACT. A nonzero locally nilpotent linear derivation δ of the polynomial algebra $K[X_d] = K[x_1, \ldots, x_d]$ in several variables over a field K of characteristic 0 is called a Weitzenböck derivation. The classical theorem of Weitzenböck states that the algebra of constants $K[X_d]^\delta$ (which coincides with the algebra of invariants of a single unipotent transformation) is finitely generated. Similarly one may consider the algebra of constants of a locally nilpotent linear derivation δ of a finitely generated (not necessarily commutative or associative) algebra which is relatively free in a variety of algebras over K. Now the algebra of constants is usually not finitely generated. Except for some trivial cases this holds for the algebra of constants $(L_d/L_d'')^\delta$ of the free metabelian Lie algebra L_d/L_d'' with d generators. We show that the vector space of the constants $(L_d'/L_d'')^\delta$ in the commutator ideal L_d'/L_d'' is a finitely generated $K[X_d]^\delta$-module. For small d, we calculate the Hilbert series of $(L_d/L_d'')^\delta$ and find the generators of the $K[X_d]^\delta$-module $(L_d'/L_d'')^\delta$. This gives also an (infinite) set of generators of the algebra $(L_d/L_d'')^\delta$.

1. Introduction

A linear operator δ of an algebra R over a field K is a derivation if $\delta(uv) = \delta(u)v + u\delta(v)$ for every $u, v \in R$. In this paper the base field K will be of characteristic 0. We fix also an integer $d \geq 2$ and a set of variables $X_d = \{x_1, \ldots, x_d\}$. Let $K[X_d] = K[x_1, \ldots, x_d]$ be the polynomial algebra in d variables. Every mapping $\delta : X_d \to K[X_d]$ can be extended in a unique way to a derivation of $K[X_d]$ which we shall denote by the same symbol δ. In our considerations δ will act as a nonzero nilpotent linear operator of the vector space KX_d with basis X_d. Such derivations are called Weitzenböck. The Jordan normal form $J(\delta)$ of the matrix of δ

$$J(\delta) = \begin{pmatrix} J_1 & 0 & \cdots & 0 \\ 0 & J_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & J_s \end{pmatrix}$$

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consists of Jordan cells with zero diagonals

$$J_i = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}, \quad i = 1, \dots, s.$$

Hence for each dimension d there is only a finite number of essentially different Weitzenböck derivations. Up to a linear change of the coordinates, the Weitzenböck derivations δ are in a one-to-one correspondence with the partition $(p_1+1, \dots, p_s+1)$ of d, where $p_1 \geq \cdots \geq p_s \geq 0$, $(p_1+1)+\cdots+(p_s+1) = d$, and the correspondence is given in terms of the size $(p_i+1)\times(p_i+1)$ of the Jordan cells J_i of $J(\delta)$, $i = 1, \dots, s$. We shall denote the derivation corresponding to this partition by $\delta(p_1, \dots, p_s)$.

Clearly, any Weitzenböck derivation δ is locally nilpotent, i.e., for any $u \in K[X_d]$ there exists an n such that $\delta^n(u) = 0$. The linear operator

$$\exp\delta = 1 + \frac{\delta}{1!} + \frac{\delta^2}{2!} + \cdots$$

acting on KX_d is unipotent. It is well known that the algebra of constants of δ

$$K[X_d]^\delta = \ker\delta = \{u \in K[X_d] \mid \delta(u) = 0\}$$

coincides with the algebra of invariants of $\exp\delta$

$$K[X_d]^{\exp\delta} = \{u \in K[X_d] \mid \exp\delta(u) = u\},$$

and the latter coincides also with the algebra of invariants $K[X_d]^{UT_2(K)}$ of the unitriangular group $UT_2(K) = \{\exp(\alpha\delta) \mid \alpha \in K\}$. This allows to study the algebra $K[X_d]^\delta$ with methods of classical invariant theory.

The classical theorem of Weitzenböck [26] states that for any Weitzenböck derivation δ the algebra of constants $K[X_d]^\delta$ is finitely generated. See the book by Nowicki [22] for more information on Weitzenböck derivations, including references and examples of explicit sets of generators of the algebra $K[X_d]^\delta$ for concrete δ. For computational aspects see also the books by Derksen and Kemper [9] and Sturmfels [25].

The polynomial algebra $K[X_d]$ is free in the class of all commutative algebras. Similarly, we may consider the relatively free algebra $F_d(\mathfrak{V})$ in a variety $\mathfrak{V}$ of (not necessarily associative) algebras. For a background on varieties of associative and Lie algebras see, respectively, the books by Drensky [11] and Bahturin [2]. As in the polynomial case, if $F_d(\mathfrak{V})$ is freely generated by the set X_d, then every map $X_d \to F_d(\mathfrak{V})$ can be extended to a derivation of $F_d(\mathfrak{V})$. Again, we shall call the derivations δ which act as nilpotent linear operators of the vector space KX_d Weitzenböck derivations and shall denote them in the same way $\delta(p_1, \dots, p_s)$ as in the polynomial case.

Drensky and Gupta [14] studied Weitzenböck derivations δ acting on relatively free associative and Lie algebras. In particular, if the polynomial identities of the variety $\mathfrak{V}$ of associative algebras follow from the identity $[x_1, x_2][x_3, x_4] = 0$ (which is equivalent to the condition that $\mathfrak{V}$ contains the algebra $U_2(K)$ of 2×2 upper triangular matrices), then the algebra of constants $F_d(\mathfrak{V})^\delta$ is not finitely generated. If $U_2(K)$ does not belong to $\mathfrak{V}$ (which implies that $\mathfrak{V}$ satisfies some Engel identity

$[x_2,\underbrace{x_1,\ldots,x_1}_{n \text{ times}}] = x_2\mathrm{ad}^n x_1 = 0)$, a result of Drensky [12] gives that the algebra $F_d(\mathfrak{V})^\delta$ is finitely generated.

Although not finitely generated in the general case, the (associative) algebra $F_d(\mathfrak{V})^\delta$ has some features typical for finitely generated (commutative) algebras. In particular, the Hilbert (or Poincaré) series of $F_d(\mathfrak{V})^\delta$ is a rational function. This follows from results of Belov [4] and Berele [6, 7] combined with ideas of classical invariant theory, see Drensky and Genov [13] and Benenati, Boumova, Drensky, Genov, and Koev [5]. Hence, it is interesting to know how far from finitely generated is the algebra $F_d(\mathfrak{V})^\delta$.

We consider this problem for free metabelian Lie algebras. Let L_d be the free Lie algebra with X_d as a set of free generators. The variety $\mathfrak{A}^2$ of metabelian (solvable of length 2) Lie algebras is defined by the polynomial identity $[[x_1,x_2],[x_3,x_4]] = 0$. The free metabelian algebra $F_d(\mathfrak{A}^2)$ is the relatively free algebra in $\mathfrak{A}^2$ and is isomorphic to the factor algebra $L_d/L_d'' = L_d/[[L_d,L_d],[L_d,L_d]]$. We denote its free generators with the same symbols x_j as the generators of L_d and $K[X_d]$. By Drensky and Gupta [14], if δ is a Weitzenböck derivation of L_d/L_d'', then the algebra of constants $(L_d/L_d'')^\delta$ is finitely generated if and only if the Jordan normal form of δ consists of one Jordan cell of size 2×2 and $d-2$ Jordan cells of size 1×1, i.e., when the rank of the matrix of δ is equal to 1.

The commutator ideal L_d'/L_d'' of the algebra L_d/L_d'' has a natural structure of a $K[X_d]$-module. Our first result is that its vector subspace $(L_d'/L_d'')^\delta$ is a finitely generated $K[X_d]^\delta$-module. Freely speaking, this means that the algebra of constants $(L_d/L_d'')^\delta$ is very close to finitely generated.

Then, using the methods of [5] we give an algorithm how to calculate the Hilbert series of $(L_d/L_d'')^\delta$ and calculate it for small d.

Let the Jordan form of δ contain a 1×1 cell. Then we may assume that δ acts as a nilpotent linear operator on KX_{d-1} and $\delta(x_d)=0$. It is well known that in the commutative case $K[X_d]^\delta = (K[X_{d-1}]^\delta)[x_d]$ and this reduces the study of the algebra $K[X_d]^\delta$ to the algebra of constants $K[X_{d-1}]^\delta$ in the polynomial algebra in $d-1$ variables. Using the methods of [5] again, we establish a similar result for the algebra of constants $(L_d/L_d'')^\delta$. The result is more complicated than in the polynomial case but we give an algorithm which expresses the generators of the $K[X_d]^\delta$-module $(L_d'/L_d'')^\delta$ in terms of the generators of the $K[X_{d-1}]^\delta$-module $(L_{d-1}'/L_{d-1}'')^\delta$ and the generators of the algebra $K[X_{d-1}]^\delta$.

Finally, we find the generators of the $K[X_d]^\delta$-module $(L_d'/L_d'')^\delta$ for $d\le 4$ and for $d=6$, $\delta=\delta(1,1,1)$. This gives also an explicit (infinite) set of generators of the algebra $(L_d/L_d'')^\delta$.

2. Finite generation

We assume that all Lie commutators are left normed, e.g.,

$$[x_1,x_2,x_3] = [[x_1,x_2],x_3] = [x_1,x_2]\mathrm{ad}x_3.$$

It is well known, see [2], that the metabelian identity implies the identity

$$[x_{j_1},x_{j_2},x_{j_{\sigma(3)}},\ldots,x_{j_{\sigma(k)}}] = [x_{j_1},x_{j_2},x_{j_3},\ldots,x_{j_k}],$$

where σ is an arbitrary permutation of $3,\dots,k$ and that L'_d/L''_d has a basis consisting of all

$$[x_{j_1},x_{j_2},x_{j_3},\dots,x_{j_k}],\quad 1\leq j_i\leq d,\quad j_1>j_2\leq j_3\leq\cdots\leq j_k.$$

Hence the polynomial algebra $K[X_d]$ acts on L'_d/L''_d by the rule

$$uf(x_1,\dots,x_d)=uf(\mathrm{ad}x_1,\dots,\mathrm{ad}x_d),\quad u\in L'_d/L''_d, f(X_d)\in K[X_d].$$

Recall the construction of abelian wreath products due to Shmel'kin [24]. Let A_d and B_d be the abelian Lie algebras with bases $\{a_1,\dots,a_d\}$ and $\{b_1,\dots,b_d\}$, respectively. Let C_d be the free right $K[X_d]$-module with free generators $a_1,\dots,a_d$. We give it the structure of a Lie algebra with trivial multiplication. The abelian wreath product $A_d\mathrm{wr}B_d$ is equal to the semidirect sum $C_d\leftthreetimes B_d$. The elements of $A_d\mathrm{wr}B_d$ are of the form $\displaystyle\sum_{j=1}^d a_jf_j(X_d)+\sum_{j=1}^d\beta_jb_j$, where f_j are polynomials in $K[X_d]$ and $\beta_j\in K$. The multiplication in $A_d\mathrm{wr}B_d$ is defined by

$$[C_d,C_d]=[B_d,B_d]=0,$$

$$[a_jf_j(X_d),b_i]=a_jf_j(X_d)x_i,\quad i,j=1,\dots,d.$$

Hence $A_d\mathrm{wr}B_d$ is a metabelian Lie algebra and every mapping $X_d\to A_d\mathrm{wr}B_d$ can be extended to a homomorphism $L_d/L''_d\to A_d\mathrm{wr}B_d$. As a special case of the embedding theorem of Shmel'kin [24], the homomorphism $\varepsilon:L_d/L''_d\to A_d\mathrm{wr}B_d$ defined by $\varepsilon(x_j)=a_j+b_j$, $j=1,\dots,d$, is a monomorphism. If

$$u=\sum[x_i,x_j]f_{ij}(X_d),\quad f_{ij}(X_d)\in K[X_d],$$

then

$$\varepsilon(u)=\sum(a_ix_j-a_jx_i)f_{ij}(X_d).$$

An element $\displaystyle\sum_{i=1}^d a_if_i(X_d)\in A_d\mathrm{wr}B_d$ is an image of an element from the commutator ideal L'_d/L''_d if and only if $\displaystyle\sum_{i=1}^d x_if_i(X_d)=0$.

If δ is a Weitzenböck derivation of L_d/L''_d, we define an action of δ on $A_d\mathrm{wr}B_d$ assuming that

$$\delta(a_j)=\sum_{i=1}^d\alpha_{ij}a_i,\quad\delta(b_j)=\sum_{i=1}^d\alpha_{ij}b_i,\quad j=1,\dots,d,$$

where $\alpha_{ij}\in K$, $i,j=1,\dots,d$, and

$$\delta(x_j)=\sum_{i=1}^d\alpha_{ij}x_i,\quad j=1,\dots,d.$$

Obviously, the vector space C_d^δ of the constants of δ in the free $K[X_d]$-module C_d is a $K[X_d]^\delta$-module. The following lemma is a partial case of [12, Proposition 3].

Lemma 2.1. *The vector space C_d^δ is a finitely generated $K[X_d]^\delta$-module.*

Clearly, if $u \in L_d/L_d''$, then $\varepsilon(\delta(u)) = \delta(\varepsilon(u))$. To simplify the notation we shall omit ε and shall think that L_d/L_d'' is a subalgebra of $A_d\mathrm{wr}B_d$. Since the action of $K[X_d]$ on L_d'/L_d'' agrees with its action on C_d, we shall also think that L_d'/L_d'' is a $K[X_d]$-submodule of C_d.

Theorem 2.2. *Let δ be a Weitzenböck derivation of the free metabelian Lie algebra L_d/L_d''. Then the vector space $(L_d'/L_d'')^\delta$ of the constants of δ in the commutator ideal L_d'/L_d'' of L_d/L_d'' is a finitely generated $K[X_d]^\delta$-module.*

Proof. By Lemma 2.1 the $K[X_d]^\delta$-module C_d^δ is finitely generated. Since the algebra $K[X_d]^\delta$ is also finitely generated, all $K[X_d]^\delta$-submodules of C_d^δ, including $(L_d'/L_d'')^\delta$, are also finitely generated. □

3. Hilbert series

Since the base field K is of characteristic 0, the relatively free algebra $F_d = F_d(\mathfrak{V})$ of the variety $\mathfrak{V}$ of (not necessarily associative or Lie) algebras is a graded vector space. If $F_d^{(n)}$ is the homogeneous component of degree n of F_d, then the Hilbert series of F_d is the formal power series

$$H(F_d, z) = \sum_{n\geq 0} \dim F_d^{(n)} z^n.$$

The algebra F_d is also multigraded, with a $\mathbb{Z}^d$-grading which counts the degree of each variable x_j in the monomials in F_d. If $F_d^{(n_1,\ldots,n_d)}$ is the multihomogeneous component of degree $(n_1, \ldots, n_d)$, then the corresponding Hilbert series of F_d is

$$H(F_d, z_1, \ldots, z_d) = \sum_{n_j\geq 0} \dim F_d^{(n_1,\ldots,n_d)} z_1^{n_1} \cdots z_d^{n_d}.$$

Similarly, if δ is a Weitzenböck derivation of F_d, the algebra of constants F_d^δ is graded and its Hilbert series is

$$H(F_d^\delta, z) = \sum_{n\geq 0} \dim(F_d^\delta)^{(n)} z^n.$$

As in the commutative case the algebra of constants F_d^δ coincides with the algebra $F_d^{UT_2(K)}$ of $UT_2(K)$-invariants, where the action of $UT_2(K)$ on F_d is defined by its realization as $UT_2(K) = \{\exp(\alpha\delta) \mid \alpha \in K\}$. There is an analogue of the integral Molien-Weyl formula due to Almkvist, Dicks and Formanek [1] which allows to calculate the Hilbert series of $F_d^{UT_2(K)}$ evaluating a multiple integral, if we know the Hilbert series $H(F_d, z_1, \ldots, z_d)$ of F_d. For varieties of associative algebras and for the variety of metabelian Lie algebras the Hilbert series of F_d is a rational function in d variables. Then the integral can be evaluated using the Residue Theorem, see the book [9] for details. Instead, in [13] and [5] another approach was suggested. It combines ideas of De Concini, Eisenbud, and Procesi [8], Berele [6, 7], and classical results of Elliott [16] and MacMahon [21]. We give a short summary of the method. For details we refer to [5].

We assume that X_d is a Jordan basis of the vector space KX_d for the Weitzenböck derivation $\delta = \delta(p_1, \ldots, p_s)$ of F_d. First we define an action of the general linear group $GL_2(K)$ on F_d. Let $Y_i = \{x_j, x_{j+1}, \ldots, x_{j+p_i}\}$ be the part of the basis X_d corresponding to the i-th Jordan cell of δ. We identify the vector space KY_i with the vector space of the binary forms (homogeneous polynomials in two commuting

variables y_{i1} and y_{i2}) of degree p_i. We assume that $GL_2(K)$ acts on the two-dimensional vector space with basis $\{y_{i1}, y_{i2}\}$ and extend its action diagonally on the polynomial algebra $K[y_{i1}, y_{i2}]$. We want to synchronize the actions on KY_i of $UT_2 = \{\exp(\alpha\delta) \mid \alpha \in K\}$ and of $UT_2(K)$ as a subgroup of $GL_2(K)$. For this purpose we define an action of the derivation δ on $K[y_{i1}, y_{i2}]$ by $\delta(y_{i1}) = 0$, $\delta(y_{i2}) = y_{i1}$. Then we identify x_{j+i_p} with $y_{i2}^{p_i}$ and $x_{j+i_p-k} = \delta^k(x_{j+i_p})$ with $\delta^k(y_{i2}^{p_i}) = p_i(p_i-1)\cdots(p_i-k+1)y_{i1}^k y_{i2}^{p_i-k}$, $k = 1, \dots, p_i$. In this way the vector space KX_d has a structure of a $GL_2(K)$-module and we extend diagonally the action of $GL_2(K)$ on the whole F_d. The basis X_d consists of eigenvectors of the diagonal subgroup of $GL_2(K)$. If $g = \xi_1 e_{11} + \xi_2 e_{22}$, $\xi_1, \xi_2 \in K^*$, is a diagonal matrix, then $g(x_{j+k}) = \xi_1^{p_i-k}\xi_2^k$, $k = 0, 1, \dots, p_i$. This defines a bigrading on F_d assuming that the bidegree of x_{j+k} is $(p_i - k, k)$. Now F_d is a direct sum of irreducible polynomial $GL_2(K)$-submodules. The irreducible polynomial $GL_2(K)$-modules are indexed by partitions $\lambda = (\lambda_1, \lambda_2)$. If $W = W(\lambda)$ is an irreducible component of F_d, it contains a unique (up to a multiplicative constant) nonzero element w of bidegree (λ_1, λ_2). It is invariant under the action of $UT_2(K)$ and by [8] the algebra of $UT_2(K)$-invariants $F_d^{UT_2(K)} = F_d^\delta$ is spanned by these vectors w. We express the Hilbert series of F_d as a bigraded vector space. For this purpose we replace in the Hilbert series $H(F_d, z_1, \dots, z_d)$ the variables $z_j, z_{j+1}, \dots, z_{j+p_i-1}, z_{j+p_i}$ corresponding to each set $Y_i = \{x_j, x_{j+1}, \dots, x_{j+p_i-1}, x_{j+p_i}\}$ by $t_1^{p_i}z, t_1^{p_i-1}t_2 z, \dots, t_1 t_2^{p_i-1}z, t_2^{p_i}z$, respectively, and obtain the Hilbert series

$$H_{GL_2}(F_d, t_1, t_2, z) = H(F_d, t_1^{p_1}z, t_1^{p_1-1}t_2 z, \dots, t_2^{p_1}z, \dots, t_1^{p_s}z, t_1^{p_s-1}t_2 z, \dots, t_2^{p_s}z).$$

The variable z gives the total degree and t_1, t_2 count the bidegree induced by the action of the diagonal subgroup of $GL_2(K)$: The coefficient of $t_1^{n_1}t_2^{n_2}z^n$ in $H_{GL_2}(F_d, t_1, t_2, z)$ is equal to the dimension of the elements of F_d which are linear combinations of products of length n in the variables X_d and are of bidegree (n_1, n_2). The Hilbert series is an infinite linear combination with nonnegative integer coefficients of Schur functions

$$H_{GL_2}(F_d, t_1, t_2, z) = \sum_{n\geq 0}\sum_{(\lambda_1,\lambda_2)} m(\lambda_1, \lambda_2, n) S_{(\lambda_1,\lambda_2)}(t_1, t_2) z^n$$

and, by the representation theory of $GL_2(K)$, the multiplicity $m(\lambda_1, \lambda_2, n)$ is equal to the multiplicity of the irreducible $GL_2(K)$-module $W(\lambda_1, \lambda_2)$ in the homogeneous component $F_d^{(n)}$ of total degree n of F_d. Hence the bigraded Hilbert series of the algebra $F_d^{UT_2(K)}$ of $UT_2(K)$-invariants is

$$H_{GL_2}(F_d^{UT_2(K)}, t_1, t_2, z) = \sum_{n\geq 0}\sum_{(\lambda_1,\lambda_2)} m(\lambda_1, \lambda_2, n) t_1^{\lambda_1}t_2^{\lambda_2}z^n$$

which is the so called multiplicity series of $H_{GL_2}(F_d, t_1, t_2, z)$ considered as a symmetric function in the variables t_1, t_2. In order to obtain the Hilbert series of $F_d^{UT_2(K)}$ as a $\mathbb{Z}$-graded vector space, it is sufficient to replace t_1 and t_2 with 1:

$$H(F_d^{UT_2(K)}, z) = \sum_{n\geq 0}(F_d^{UT_2(K)})^{(n)}z^n = H_{GL_2}(F_d^{UT_2(K)}, 1, 1, z).$$

To determine the multiplicity series of $H_{GL_2}(F_d, t_1, t_2, z)$ we follow the receipt of [6, 13, 5]. We consider the function

$$f(t_1, t_2, z) = (t_1 - t_2)H_{GL_2}(F_d, t_1, t_2, z)$$

which is skewsymmetric in t_1 and t_2 and consider the Laurent series

$$f(t_1\xi, t_2/\xi, z) = \sum_{k=-\infty}^{+\infty} f_k(t_1, t_2, z)\xi^k.$$

Then

$$H_{GL_2}(F_d^{UT_2(K)}, t_1, t_2, z) = \frac{1}{t_1}\sum_{k\geq 0} f_k(t_1, t_2, z).$$

By the theorem of Belov [4] the Hilbert series of the relatively free associative algebra F_d is a rational function with denominator which is a product of factors of the form $1 - z_1^{q_1}\cdots z_d^{q_d}$. Berele [6, 7] calls such rational functions nice and proves that the multiplicity series of a nice rational symmetric function is nice again. The Hilbert series of the free metabelian Lie algebra L_d/L_d'' is also nice, see below. By [5], when $H_{GL_2}(F_d, t_1, t_2, z)$ is a nice rational function, its multiplicity series (which is equal to $H_{GL_2}(F_d^{UT_2(K)}, t_1, t_2, z)$) can be evaluated by the method of Elliott [16] and its further development by McMahon [21], the so called partition analysis or Ω-calculus. In [5] an improvement of the Ω-calculus is used, in the spirit of the algorithm of Xin [27] which involves partial fractions and allows to perform computations with standard functions of Maple on a usual personal computer.

The next fact is well known, see, e.g., [10].

Lemma 3.1. *The Hilbert series of the free metabelian Lie algebra L_d/L_d'' is*

$$H(L_d/L_d'', z_1, \ldots, z_d) = 1 + (z_1 + \cdots + z_d) + (z_1 + \cdots + z_d - 1)\prod_{j=1}^{d}\frac{1}{1 - z_j}.$$

Now we shall give the Hilbert series of the subalgebras of constants of Weitzenböck derivations of free metabelian Lie algebras with small number of generators. In some of the cases we give both Hilbert series, as graded and bigraded vector spaces, because we shall use the results in the last section of our paper. We do not give results for derivations with a one-dimensional Jordan cell because we shall handle them in the next section.

Example 3.2. Let $\delta = \delta(p_1, \ldots, p_s)$ be the Weitzenböck derivation of the free metabelian Lie algebra L_d/L_d'' which has Jordan cells of size $p_1 + 1, \ldots, p_s + 1$. Then the Hilbert series of the algebra of constants $(L_d/L_d'')^\delta$ are:
$d = 2$, $\delta = \delta(1)$:

$$H_{GL_2}((L_2/L_2'')^\delta, t_1, t_2, z) = t_1 z + \frac{t_1 t_2 z^2}{1 - t_1 z}, \quad H((L_2/L_2'')^\delta, z) = z + \frac{z^2}{1 - z};$$

$d = 3$, $\delta = \delta(2)$:

$$H_{GL_2}((L_3/L_3'')^\delta, t_1, t_2, z) = t_1^2 z + \frac{t_1^3 t_2 z^2}{(1 - t_1^2 z)(1 - t_1 t_2 z)},$$

$$H((L_3/L_3'')^\delta, z) = z + \frac{z^2}{(1 - z)^2};$$

$d = 4$, $\delta = \delta(3)$:

$$H_{GL_2}((L_4/L_4'')^\delta, t_1, t_2, z) = t_1^3 z + \frac{t_1^3 t_2 z^2 (t_1^2 + t_2^2 + t_1^4 t_2^4 z^2 + t_1^5 t_2^6 z^3 - t_1^8 t_2^6 z^4)}{(1 - t_1^3 z)(1 - t_1^2 t_2 z)(1 - t_1^6 t_2^6 z^4)},$$

$$H((L_4/L_4'')^\delta,z)=z+\frac{z^2(2+z^2+z^3-z^4)}{(1-z)^2(1-z^4)};$$

$d=4$, $\delta=\delta(1,1)$:

$$H_{GL_2}((L_4/L_4'')^\delta,t_1,t_2,z)=2t_1z+\frac{t_1z^2(t_1+3t_2-t_1^2t_2z^2)}{(1-t_1z)^2(1-t_1t_2z^2)},$$

$$H((L_4/L_4'')^\delta,z)=2z+\frac{z^2(4-z^2)}{(1-z)^2(1-z^2)};$$

$d=5$, $\delta=\delta(4)$:

$$H((L_5/L_5'')^\delta,z)=z+\frac{z^2(2+2z+z^2-2z^4+z^5)}{(1-z)^2(1-z^2)(1-z^3)};$$

$d=5$, $\delta=\delta(2,1)$:

$$H((L_5/L_5'')^\delta,z)=2z+\frac{z^2(4+2z^2-3z^3+z^4)}{(1-z)^3(1-z^3)};$$

$d=6$, $\delta=\delta(5)$:

$$H((L_6/L_6'')^\delta,z)=z+\frac{p(z)}{(1-z)^2(1-z^4)(1-z^6)(1-z^8)},$$

$$p(z)=z^2(3+3z+7z^2+10z^3+11z^4+14z^5+13z^6+16z^7+12z^8$$

$$+8z^9+10z^{10}+3z^{11}+5z^{12}-z^{13}+z^{14}-z^{16}+2z^{17}-z^{18});$$

$d=6$, $\delta=\delta(3,1)$:

$$H((L_6/L_6'')^\delta,z)=2z+\frac{z^2(5+6z+8z^2+11z^3+5z^4-2z^5+3z^6-2z^7+2z^9-z^{10})}{(1-z)^2(1-z^2)(1-z^4)^2};$$

$d=6$, $\delta=\delta(2,2)$:

$$H((L_6/L_6'')^\delta,z)=2z+\frac{z^2(5+8z-6z^3+2z^4+2z^5-z^6)}{(1-z)^2(1-z^2)^3};$$

$d=6$, $\delta=\delta(1,1,1)$:

$$H_{GL_2}((L_6/L_6'')^\delta,t_1,t_2,z)=3t_1z+\frac{t_1z^2p(t_1,t_2,z)}{(1-t_1z)^3(1-t_1t_2z^2)^3},$$

$$p=3(t_1+2t_2)+t_1(-t_1+t_2)z-9t_1^2t_2z^2+3t_1^2t_2(-3t_2+t_1)z^3$$

$$+t_1^2t_2^2(9t_1-t_2)z^4+3t_1^3t_2^2(t_2-t_1)z^5-3t_1^4t_2^3z^6+t_1^5t_2^3z^7),$$

$$H((L_6/L_6'')^\delta,z)=3z+\frac{z^2(9+9z-6z^3+2z^4+2z^5-z^6)}{(1-z)^2(1-z^2)^3}.$$

4. Derivations with one-dimensional Jordan cell

In this section we assume that the Jordan form of δ contains a 1×1 cell, δ acts as a nilpotent linear operator on KX_{d-1}, and $\delta(x_d)=0$. We fix a finite system $\{f_1(X_{d-1}),\ldots,f_l(X_{d-1})\}$ of generators of the algebra of constants $K[X_{d-1}]^{\delta}$. Without loss of generality we may assume that the polynomials $f_r(X_{d-1})$ are homogeneous, $r=1,\ldots,l$. Also, we fix a system $\{c_1,\ldots,c_k\}$ of generators of the $K[X_{d-1}]^{\delta}$-module $(L'_{d-1}/L''_{d-1})^{\delta}$. Our purpose is to find a generating set of the $K[X_d]^{\delta}$-module $(L'_d/L''_d)^{\delta}$.

Lemma 4.1. *The Hilbert series of $(L'_d/L''_d)^{\delta}$, $(L'_{d-1}/L''_{d-1})^{\delta}$, and $K[X_{d-1}]^{\delta}$ are related by*

$$H_{GL_2}((L'_d/L''_d)^{\delta},t_1,t_2,z)=\frac{1}{1-z}H_{GL_2}((L'_{d-1}/L''_{d-1})^{\delta},t_1,t_2,z)$$
$$+\frac{z}{1-z}(H_{GL_2}(K[X_{d-1}]^{\delta},t_1,t_2,z)-1),$$
$$H((L'_d/L''_d)^{\delta},z)=\frac{1}{1-z}(H((L'_{d-1}/L''_{d-1})^{\delta},z)+z(H(K[X_{d-1}]^{\delta},z)-1)).$$

Proof. Let δ act on KX_{d-1} as $\delta=\delta(p_1,\ldots,p_{s-1})$. Then it acts on KX_d as $\delta(p_1,\ldots,p_{s-1},0)$. By Lemma 3.1 the Hilbert series of the commutator ideal of L_d/L''_d is

$$H(L'_d/L''_d,z_1,\ldots,z_d)=1+(z_1+\cdots+z_d-1)\prod_{j=1}^{d}\frac{1}{1-z_j}.$$

(In the Hilbert series $H(L_d/L''_d,z_1,\ldots,z_d)$ in the lemma we remove the summand $z_1+\cdots+z_d$ which gives the contribution of the elements of first degree.) Following the procedure described in Section 3 we replace its variables z_j with $t_1^{q_j}t_2^{r_j}z$, where the nonnegative integers q_j,r_j depend on the size of the corresponding Jordan cell and the position of the variable x_j in the Jordan basis of KX_d. In particular, we have to replace the variable z_n with z. Hence

$$H_{GL_2}(L'_d/L''_d,t_1,t_2,z)=1+\left(\sum_{j=1}^{d-1}t_1^{q_j}t_2^{r_j}z+z-1\right)\frac{1}{1-z}\prod_{j=1}^{d-1}\frac{1}{1-t_1^{q_j}t_2^{r_j}z}$$
$$=\frac{1}{1-z}\left(\left(1+\left(\sum_{j=1}^{d-1}t_1^{q_j}t_2^{r_j}z-1\right)\prod_{j=1}^{d-1}\frac{1}{1-t_1^{q_j}t_2^{r_j}z}\right)+z\left(\prod_{j=1}^{d-1}\frac{1}{1-t_1^{q_j}t_2^{r_j}z}-1\right)\right)$$
$$=\frac{1}{1-z}H_{GL_2}(L'_{d-1}/L''_{d-1},t_1,t_2,z)+\frac{z}{1-z}(H_{GL_2}(K[X_{d-1}],t_1,t_2,z)-1).$$

The Hilbert series $H_{GL_2}((L'_d/L''_d)^{\delta},t_1,t_2,z)$ is equal to the multiplicity series of $H_{GL_2}(L'_d/L''_d,t_1,t_2,z)$. Similar statements hold for the other two Hilbert series $H_{GL_2}((L'_{d-1}/L''_{d-1})^{\delta},t_1,t_2,z)$ and $H_{GL_2}(K[X_{d-1}]^{\delta},t_1,t_2,z)$. Hence

$$H_{GL_2}((L'_d/L''_d)^{\delta},t_1,t_2,z)=\frac{1}{1-z}H_{GL_2}((L'_{d-1}/L''_{d-1})^{\delta},t_1,t_2,z)$$
$$+\frac{z}{1-z}(H_{GL_2}(K[X_{d-1}]^{\delta},t_1,t_2,z)-1)$$

which implies that

$$H((L'_d/L''_d)^\delta, z) = \frac{1}{1-z} H((L'_{d-1}/L''_{d-1})^\delta, z) + \frac{z}{1-z}(H(K[X_{d-1}]^\delta, z) - 1).$$

□

Let $\omega = \omega(K[X_{d-1}])$ be the augmentation ideal of $K[X_{d-1}]$, i.e., the ideal of all polynomials without constant term. We define a K-linear map

$$\pi : \omega(K[X_{d-1}]) \to L'_d/L''_d$$

by

$$\pi(x_{j_1} \cdots x_{j_n}) = \sum_{k=1}^{n} [x_d, x_{j_k}] x_{j_1} \cdots x_{j_{k-1}} x_{j_{k+1}} \cdots x_{j_n}, \quad x_{j_1} \cdots x_{j_n} \in K[X_{d-1}], n \geq 1.$$

Lemma 4.2. (i) *The map π satisfies the equality*

$$\pi(uv) = \pi(u)v + \pi(v)u, \quad u, v \in \omega.$$

(ii) *The derivation δ and the map π commute.*

Proof. (i) It is sufficient to show the equality for u, v being monomials only. Let $u = x_{i_1} \cdots x_{i_m}$ and $v = x_{j_1} \cdots x_{j_n}$. We use the standard notation $x_{j_1} \cdots \widehat{x_{j_k}} \cdots x_{j_n}$ to indicate that x_{j_k} does not participate in the product. Then

$$\pi(uv) = \pi(x_{i_1} \cdots x_{i_m} x_{j_1} \cdots x_{j_n}) = \left(\sum_{l=1}^{m} [x_d, x_{i_l}] x_{i_1} \cdots \widehat{x_{i_l}} \cdots x_{i_m} \right) (x_{j_1} \cdots x_{j_n})$$

$$+ \left(\sum_{k=1}^{n} [x_d, x_{j_k}] x_{j_1} \cdots \widehat{x_{j_k}} \cdots x_{j_n} \right) (x_{i_1} \cdots x_{i_m}) = \pi(u)v + \pi(v)u.$$

(ii) Again, it is sufficient to show the equality for monomials only. We proceed by induction on the length of the monomials. If $u = x_j$ and

$$\delta(x_j) = \sum_{i=1}^{d-1} \alpha_{ij} x_i, \quad \alpha_{ij} \in K, j = 1, \ldots, d-1,$$

then

$$\pi(\delta(x_j)) = \sum_{i=1}^{d-1} \alpha_{ij} \pi(x_i) = \sum_{i=1}^{d-1} \alpha_{ij} [x_d, x_i] = [x_d, \sum_{i=1}^{d-1} \alpha_{ij}(x_i)] = \delta(\pi(x_j))$$

because $\delta(x_d) = 0$. Let the monomials u and v belong to ω. Using that δ is a derivation of L_d/L''_d which by the inductive arguments commute with π on u and v, and applying (i), we obtain

$$\delta(\pi(uv)) = \delta(\pi(u)v + \pi(v)u) = \delta(\pi(u))v + \pi(u)\delta(v) + \delta(\pi(v))u + \pi(v)\delta(u)$$

$$= \pi(\delta(u))v + \pi(u)\delta(v) + \pi(\delta(v))u + \pi(v)\delta(u) = \pi(\delta(u)).$$

□

The next theorem and its corollary are the main results of the section.

Theorem 4.3. *Let X_d be a Jordan basis of the derivation δ acting on KX_d and let δ have a 1×1 Jordan cell corresponding to x_d. Let $\{v_i \mid i\in I\}$ and $\{u_j \mid j\in J\}$ be, respectively, homogeneous bases of $(L'_{d-1}/L''_{d-1})^{\delta}$ and $\omega(K[X_{d-1}])^{\delta}$ with respect to both $\mathbb{Z}$- and $\mathbb{Z}^2$-gradings. Then $(L'_d/L''_d)^{\delta}$ has a basis*

$$\{v_ix_d^n, \pi(u_j)x_d^n \mid i\in I, j\in J, n\geq 0\}.$$

Proof. The Hilbert series of $(L'_{d-1}/L''_{d-1})^{\delta}$ and $\omega(K[X_{d-1}])^{\delta}$ are equal, respectively, to the generating functions of their bases. Hence

$$H_{GL_2}((L'_{d-1}/L''_{d-1})^{\delta},t_1,t_2,z)=\sum_{i\in I}t_1^{q_i}t_2^{r_i}z^{m_i},$$

where v_i is of bidegree (p_i,q_i) and of total degree m_i. Since x_d is of bidegree $(0,0)$ and of total degree 1, the generating function of the set $V=\{v_ix_d^n \mid i\in I, n\geq 0\}$ is

$$G(V,t_1,t_2,z)=\sum_{n\geq 0}\sum_{i\in I}t_1^{q_i}t_2^{r_i}z^{m_i}z^n=\frac{1}{1-z}H_{GL_2}((L'_{d-1}/L''_{d-1})^{\delta},t_1,t_2,z).$$

The map π sends the monomials of $\omega(K[X_{d-1}])$ to linear combinations of commutators with an extra variable x_d in the beginning of each commutator. Hence, if the Hilbert series of $\omega(K[X_{d-1}])^{\delta}$ is

$$H_{GL_2}(\omega(K[X_{d-1}])^{\delta},t_1,t_2,z)=H_{GL_2}(K[X_{d-1}]^{\delta},t_1,t_2,z)-1=\sum_{n\geq 0}\sum_{j\in J}t_1^{k_j}t_2^{l_j}z^{n_j},$$

where the bidegree of u_j is (k_j,l_j) and its total degree is n_j, then the generating function of the set $U=\{\pi(u_j)x_d^n \mid j\in J, n\geq 0\}$ is

$$G(U,t_1,t_2,z)=\sum_{n\geq 0}\sum_{j\in J}t_1^{k_j}t_2^{l_j}z^{n_i+1}z^n=\frac{z}{1-z}(H_{GL_2}(K[X_{d-1}]^{\delta},t_1,t_2,z)-1).$$

Hence, by Lemma 4.1

$$H_{GL_2}((L'_d/L''_d)^{\delta},t_1,t_2,z)=G(V,t_1,t_2,z)+G(U,t_1,t_2,z).$$

Since both sets V and U are contained in $(L'_d/L''_d)^{\delta}$, we shall conclude that $V\cup U$ is a basis of $(L'_d/L''_d)^{\delta}$ if we show that the elements of $V\cup U$ are linearly independent. For this purpose it is more convenient to work in the abelian wreath product $A_d\mathrm{wr}B_d$. The elements v_i belong to $L'_{d-1}/L''_{d-1}\subset A_d\mathrm{wr}B_d$ and hence are of the form

$$v_i=\sum_{k=1}^{d-1}a_kg_{ki}(X_{d-1}),\quad g_{ki}(X_{d-1})\in K[X_{d-1}].$$

Hence

$$v_ix_d^n=\sum_{k=1}^{d-1}a_kg_{ki}(X_{d-1})x_d^n$$

On the other hand, the elements $\pi(u_j)$ are of the form

$$\pi(u_j)=\sum_{k=1}^{d-1}[x_d,x_k]h_{kj}(X_{d-1})=n_ja_d\sum_{k=1}^{d-1}x_kh_{kj}(X_{d-1})-\sum_{k=1}^{d-1}a_kh_{kj}(X_{d-1})x_d$$

$$=n_ja_du_j-\sum_{k=1}^{d-1}a_kh_{kj}(X_{d-1})x_d.$$

Hence

$$\pi(u_j)x^n = (n_j a_d u_j - \sum_{k=1}^{d-1} a_k h_{kj}(X_{d-1})x_d)x_d^n.$$

Let $v = \sum \xi_{in} v_i x_d^n + \sum \eta_{jn}\pi(u_j)x^n = 0$ for some $\xi_{in}, \eta_{jn} \in K$. Since the elements u_j are linearly independent in $\omega(K[X_{d-1}])^\delta$, comparing the coefficient of a_d in v we conclude that $\eta_{jn} = 0$. Then, using that the elements v_i are linearly independent in $(L'_{d-1}/L''_{d-1})^\delta$, we derive that $\xi_{in} = 0$. Hence the set $V \cup U$ is a basis of $(L'_d/L''_d)^\delta$. $\square$

Corollary 4.4. *Let X_d be a Jordan basis of the derivation δ acting on KX_d and let δ have a 1×1 Jordan cell corresponding to x_d. Let $\{c_1, \dots, c_k\}$ and $\{f_1, \dots, f_l\}$ be, respectively, homogeneous generating sets of the $K[X_{d-1}]^\delta$-module $(L'_{d-1}/L''_{d-1})^\delta$ and of the algebra of constants $K[X_{d-1}]^\delta$. Then the $K[X_d]^\delta$-module $(L'_d/L''_d)^\delta$ is generated by the set $\{c_1, \dots, c_k\} \cup \{\pi(f_1), \dots, \pi(f_l)\}$.*

Proof. Clearly, the $K[X_{d-1}]^\delta$-module $(L'_{d-1}/L''_{d-1})^\delta$ is spanned by the elements $c_j f_1^{q_1} \cdots f_l^{q_l}$. In particular, in this way we obtain all elements v_j from the basis of the vector space $(L'_{d-1}/L''_{d-1})^\delta$. Since $x_d \in K[X_d]^\delta$, we obtain also all elements $v_j x^n$. By Lemma 4.2 (i) we obtain that the $K[X_{d-1}]^\delta$-module $\pi(\omega(K[X_{d-1}])^\delta)$ is generated by $\pi(f_1), \dots, \pi(f_l)$. Hence all elements $\pi(u_j)$, where $\{u_j \mid j \in J\}$ is the basis of $\omega(K[X_{d-1}])^\delta$, belong to the $K[X_{d-1}]^\delta$-module generated by $\pi(f_1), \dots, \pi(f_l)$. In this way we obtain also the elements $\pi(u_j)x_d^n$ and derive that $\{c_1, \dots, c_k\} \cup \{\pi(f_1), \dots, \pi(f_l)\}$ generate the $K[X_d]^\delta$-module $(L'_d/L''_d)^\delta$. $\square$

Example 4.5. Let $d = 4$ and let the Jordan normal form of δ have two cells, of size 3×3 and 1×1, respectively. Hence $\delta = \delta(2, 0)$ in our notation. By Example 5.1 for $d = 3$ and $\delta = \delta(2)$, the algebra $K[X_3]^\delta$ is generated by $f_1 = x_1$ and $f_2 = x_2^2 - 2x_1x_3$. The $K[X_3]^\delta$-module $(L'_3/L''_3)^\delta$ is generated by $c_1 = [x_2, x_1]$ and $c_2 = [x_3, x_1, x_1] - [x_2, x_1, x_2]$. Hence, by Corollary 4.4, the $K[X_4]^\delta$-module $(L'_4/L''_4)^\delta$ is generated by c_1, c_2 and

$$\pi(f_1) = [x_4, x_1], \quad \pi(f_2) = 2([x_4, x_2, x_2] - [x_4, x_1, x_3] - [x_4, x_3, x_1]).$$

5. Generating sets for small number of generators

In this section we shall find the generators of the $K[X_d]^\delta$-module $(L'_d/L''_d)^\delta$ for $d \leq 4$ and for $d = 6$, $\delta = \delta(1, 1, 1)$. By Corollary 4.4, we shall assume that δ has no 1×1 Jordan cells.

Example 5.1. Let $d = 3$, $\delta = \delta(2)$, and let $\delta(x_1) = 0$, $\delta(x_2) = x_1$, $\delta(x_3) = x_2$. It is well known, see e.g., [22], that $K[X_3]^\delta$ is generated by the algebraically independent polynomials $f_1 = x_1$, $f_2 = x_2^2 - 2x_1x_3$. Hence

$$H_{GL_2}(K[X_3]^\delta, t_1, t_2, z) = \frac{1}{(1 - t_1^2 z)(1 - t_1^2 t_2^2 z^2)}.$$

By Example 3.2

$$H_{GL_2}((L'_3/L''_3)^\delta, t_1, t_2, z) = \frac{t_1^3 t_2 z^2}{(1 - t_1^2 z)(1 - t_1 t_2 z)} = \frac{t_1^3 t_2 z^2 (1 + t_1 t_2 z)}{(1 - t_1^2 z)(1 - t_1^2 t_2^2 z^2)}.$$

It is easy to see that the Lie elements

$$c_1 = [x_2, x_1], \quad c_2 = [x_3, x_1, x_1] - [x_2, x_1, x_2]$$

belong to $(L'_3/L''_3)^\delta$ and are of bidegree $(3,1)$ and $(4,2)$, respectively. If c_1 and c_2 generate a free $K[X_3]^\delta$-submodule of $(L'_3/L''_3)^\delta$, its Hilbert series is

$$(t_1^3t_2z^2+t_1^4t_2^2z^3)H_{GL_2}(K[X_3]^\delta,t_1,t_2,z)=\frac{t_1^3t_2z^2(1+t_1t_2z)}{(1-t_1^2z)(1-t_1^2t_2^2z^2)}$$
$$=H_{GL_2}((L'_3/L''_3)^\delta,t_1,t_2,z).$$

Then we can derive that c_1 and c_2 generate the whole $K[X_3]^\delta$-module $(L'_3/L''_3)^\delta$. Hence it is sufficient to show that c_1 and c_2 generate a free $K[X_3]^\delta$-module. Let $c_1u_1(f_1,f_2)+c_2u_2(f_1,f_2)=0$ for some $u_1(f_1,f_2),u_2(f_1,f_2)\in K[f_1,f_2]$. Working in the wreath product $A_3\mathrm{wr}B_3$ we obtain

$$0=(a_2x_1-a_1x_2)u_1(f_1,f_2)+((a_3x_1-a_1x_3)x_1-(a_2x_1-a_1x_2)x_2)u_2(f_1,f_2)$$
$$=a_1(-x_2u_1(f_1,f_2)+(-x_1x_3+x_2^2)u_2(f_1,f_2))$$
$$+a_2(x_1u_1(f_1,f_2)-x_1x_2u_2(f_1,f_2))+a_3x_1^2u_2(f_1,f_2).$$

Since the coefficient $x_1^2u_2(f_1,f_2)$ of a_3 is equal to 0, we obtain that $u_2(f_1,f_2)=0$. Similarly, the coefficient of a_1 gives that $u_1(f_1,f_2)=0$ and this shows that the $K[X_3]^\delta$-module $(L'_3/L''_3)^\delta$ is generated by c_1,c_2. As a vector space $(L_3/L''_3)^\delta$ is spanned by the elements x_1, $c_1f_1^{q_1}f_2^{r_1}$, and $c_2f_1^{q_2}f_2^{r_2}$, $q_j,r_j\geq 0$. This easily implies that the algebra $(L_3/L''_3)^\delta$ is generated by the infinite set

$$\{x_1,c_1f_2^{r_1},c_2f_2^{r_2}\mid r_j\geq 0\}.$$

Example 5.2. Let $d=4$, $\delta=\delta(3)$, and let $\delta(x_1)=0$, $\delta(x_2)=x_1$, $\delta(x_3)=x_2$, $\delta(x_4)=x_3$. Then, see [22], $K[X_4]^\delta$ is generated by

$$f_1=x_1,\quad f_2=x_2^2-2x_1x_3,\quad f_3=x_2^3-3x_1x_2x_3+3x_1^2x_4,$$
$$f_4=x_2^2x_3^2-2x_2^3x_4+6x_1x_2x_3x_4-\frac{8}{3}x_1x_3^3-3x_1^2x_4^2.$$

The generators of $K[X_4]^\delta$ satisfy the defining relation

$$f_3^2=f_2^3-3f_1^2f_4$$

and the algebra $K[X_4]^\delta$ has the presentation

$$K[X_4]^\delta\cong K[f_1,f_2,f_3,f_4\mid f_3^2=f_2^3-3f_1^2f_4].$$

In particular, as a vector space $K[X_4]^\delta$ has a basis

$$\{f_1^{q_1}f_2^{q_2}f_4^{q_4},f_1^{q_1}f_2^{q_2}f_3f_4^{q_4}\mid q_1,q_2,q_4\geq 0\}$$

and its Hilbert series is

$$H_{GL_2}(K[X_4]^\delta,t_1,t_2,z)=\frac{1+t_1^6t_2^3z^3}{(1-t_1^3z)(1-t_1^4t_2^2z^2)(1-t_1^6t_2^6z^4)}.$$

By Example 3.2 the Hilbert series of $(L'_4/L''_4)^\delta$ is

$$H_{GL_2}((L'_4/L''_4)^\delta,t_1,t_2,z)=\frac{t_1^3t_2z^2(t_1^2+t_2^2+t_1^4t_2^4z^2+t_1^5t_2^6z^3-t_1^8t_2^6z^4)}{(1-t_1^3z)(1-t_1^2t_2z)(1-t_1^6t_2^6z^4)}$$
$$=\frac{t_1^3t_2z^2(t_1^2+t_2^2+t_1^4t_2^4z^2+t_1^5t_2^6z^3-t_1^8t_2^6z^4)(1+t_1^2t_2z)}{(1-t_1^3z)(1-t_1^4t_2^2z^2)(1-t_1^6t_2^6z^4)}$$
$$=(t_1^5t_2+t_1^3t_2^3)z^2(1+t_1^3z)+(t_1^7t_2^2+t_1^5t_2^4)z^3+\cdots$$

This suggests that the $K[X_4]^\delta$-module $(L'_4/L''_4)^\delta$ has two generators c_1 and c_2 of bidegree $(5,1)$ and $(3,3)$, respectively. They together with c_1f_1 and c_2f_1 give the contribution $(t_1^5t_2+t_1^3t_2^3)z^2(1+t_1^3z)$. We also expect two generators c_3 and c_4 of

bidegree $(7,2)$ and $(5,4)$, respectively. By easy calculations we have found the explicit form of c_1, c_2, c_3, c_4:

$$c_1 = [x_2, x_1], \quad c_2 = [x_4, x_1] - [x_3, x_2],$$

$$c_3 = [x_3, x_1, x_1] - [x_2, x_1, x_2], \quad c_4 = 3[x_2, x_1, x_4] - 2[x_3, x_1, x_3] + [x_3, x_2, x_2].$$

For example, c_4 is a linear combination of all commutators of degree 3 and bidegree $(5,4)$: $[x_2, x_1, x_4]$, $[x_4, x_1, x_2]$, $[x_3, x_1, x_3]$, and $[x_3, x_2, x_2]$:

$$c_4 = \gamma_1[x_2, x_1, x_4] + \gamma_2[x_4, x_1, x_2] + \gamma_3[x_3, x_1, x_3] + \gamma_4[x_3, x_2, x_2], \quad \gamma_1, \gamma_2, \gamma_3, \gamma_4 \in K,$$

and the condition $\delta(c_4) = 0$ gives

$$0 = \gamma_1[x_2, x_1, x_3] + \gamma_2([x_3, x_1, x_2] + [x_4, x_1, x_1])$$
$$+\gamma_3([x_2, x_1, x_3] + [x_3, x_1, x_2]) + \gamma_4([x_3, x_1, x_2] + [x_3, x_2, x_1])$$
$$= (\gamma_1 + \gamma_3 - \gamma_4)[x_2, x_1, x_3] + (\gamma_2 + \gamma_3 + 2\gamma_4)[x_3, x_1, x_2] + \gamma_2[x_4, x_1, x_1],$$

Hence

$$\gamma_1 + \gamma_3 - \gamma_4 = \gamma_2 + \gamma_3 + 2\gamma_4 = \gamma_2 = 0$$

and, up to a multiplicative constant, the only solution is

$$\gamma_1 = 3, \quad \gamma_2 = 0, \quad \gamma_3 = -2, \quad \gamma_4 = 0.$$

Similarly, we obtain one more generator of bidegree $(7,5)$:

$$c_5 = 3(-[x_3, x_1, x_1, x_4] + [x_2, x_1, x_2, x_4] + [x_3, x_1, x_2, x_3])$$
$$-4[x_2, x_1, x_3, x_3] - [x_3, x_2, x_2, x_2].$$

The Hilbert series of the free $K[X_4]^\delta$-module generated by five elements of bidegree $(5,1)$, $(3,3)$, $(7,2)$, $(5,4)$, and $(7,5)$ is

$$H_{GL_2}(t_1, t_2, z) = \frac{t_1^3 t_2 z^2((1 + t_1^2 t_2 z)(t_1^2 + t_2^2) + t_1^4 t_2^4 z^2)(1 + t_1^6 t_2^3 z^3)}{(1 - t_1^3 z)(1 - t_1^4 t_2^2 z^2)(1 - t_1^6 t_2^6 z^4)}.$$

Hence

$$H_{GL_2}(t_1, t_2, z) - H_{GL_2}((L_4'/L_4'')^\delta, t_1, t_2, z) = (t_1^3 - t_2^3) t_1^8 t_2^4 z^5 + \cdots$$

which suggests that there is a relation of bidegree $(11,4)$ and a generator of bidegree $(8,7)$. Continuing in the same way, we have found the generators

$$c_6 = -9[x_2, x_1, x_1, x_4, x_4] + 18[x_3, x_1, x_1, x_3, x_4] - 12[x_4, x_1, x_1, x_3, x_3]$$
$$-9[x_3, x_1, x_2, x_2, x_4] + 12[x_4, x_1, x_2, x_2, x_3] + 4[x_2, x_1, x_3, x_3, x_3]$$
$$-6[x_3, x_1, x_2, x_3, x_3] - 3[x_4, x_2, x_2, x_2, x_2] + 3[x_3, x_2, x_2, x_2, x_3]$$

$$c_7 = -18[x_3, x_1, x_1, x_1, x_4, x_4] + 18[x_4, x_1, x_1, x_1, x_3, x_4] + 18[x_2, x_1, x_1, x_2, x_4, x_4]$$
$$-9[x_4, x_1, x_1, x_2, x_2, x_4] - 18[x_2, x_1, x_1, x_3, x_3, x_4] + 18[x_3, x_1, x_1, x_2, x_3, x_4]$$
$$-18[x_4, x_1, x_1, x_2, x_3, x_3] + 8[x_3, x_1, x_1, x_3, x_3, x_3] - 9[x_2, x_1, x_2, x_2, x_3, x_4]$$
$$-3[x_3, x_1, x_2, x_2, x_2, x_4] + 15[x_4, x_1, x_2, x_2, x_2, x_3] + 10[x_2, x_1, x_2, x_3, x_3, x_3]$$
$$-12[x_3, x_1, x_2, x_2, x_3, x_3] - 3[x_4, x_2, x_2, x_2, x_2, x_2] + 3[x_3, x_2, x_2, x_2, x_2, x_3]$$

of bidegree $(8,7)$ and $(10,8)$, respectively. We have also found the relations

$$R_1(11,4) : c_1 f_3 = -c_3 f_2 + c_4 f_1^2,$$
$$R_2(13,5) : c_3 f_3 = -(c_1 f_2^2 + c_5 f_1^2),$$
$$R_3(11,7) : c_4 f_3 = -(3c_1 f_4 + c_5 f_2),$$
$$R_4(11,7) : c_6 f_1 = 3(c_1 f_4 - c_2 f_2^2 + c_5 f_2),$$

$$R_5(13,8): c_5f_3 = 3c_3f_4 - c_4f_2^2,$$
$$R_6(13,8): c_7f_1 = 3(-c_2f_2f_3 + 2c_3f_4 - c_4f_2^2),$$
$$R_7(14,10): c_6f_3 = 3c_4f_1f_4 + c_7f_2,$$
$$R_8(16,11): c_7f_3 = 9c_2f_1f_2f_4 - 6c_5f_1f_4 + c_6f_2^2.$$

The above relations show that c_jf_3 can be replaced with a linear combination of other generators if $j \neq 2$. Similarly for c_6f_1 and c_7f_1. Hence the $K[X_d]^\delta$-module generated by $c_1,\dots,c_7$ is spanned by

$$C = \{c_jf_1^{q_j}f_2^{r_j}f_4^{s_j} \mid q_j,r_j,s_j \geq 0, j = 1,3,4,5\}$$
$$\cup\{c_jf_2^{r_j}f_4^{s_j} \mid r_j,s_j \geq 0, j = 6,7\} \cup \{c_2f_1^{q_2}f_2^{r_2}f_3^{\varepsilon}f_4^{s_2} \mid q_2,r_2,s_2 \geq 0, \varepsilon = 0,1\}.$$

It is easy to check that the generating function of the set C is equal to the Hilbert series of $(L'_d/L''_d)^\delta$. Hence, if we show that the elements of C are linearly independent, we shall conclude that the $K[X_d]^\delta$-module $(L'_d/L''_d)^\delta$ is generated by $c_1,\dots,c_7$. Let

$$\sum_{j=1}^{7} c_ju_j + c_2f_3u_8 = 0,$$

where u_j are polynomials in f_1,f_2,f_4, $j = 1,\dots,8$, and u_6,u_7 do not depend on f_1. We shall show that this implies that $u_j = 0$, $j = 1,\dots,8$. We shall work in the abelian wreath product $A_4\mathrm{wr}B_4$ and shall denote by v_i the coordinate of a_i of $v \in A_4\mathrm{wr}B_4$. The four coordinates v_i of

$$v = \sum_{j=1}^{7} c_ju_j + c_2f_3u_8 = \sum_{i=1}^{4} a_iv_i = 0$$

define a linear homogeneous system

$$v_i = 0, \quad i = 1,\dots,4,$$

with unknowns $u_1,\dots,u_8$. First, we substitute $x_2 = 0$. Then f_1,f_2,f_4 become

$$\bar{f}_1 = x_1, \quad \bar{f}_2 = -2x_1x_3, \quad \bar{f}_4 = -\frac{8}{3}x_1x_3^3 - 3x_1^2x_4^2.$$

Similarly, c_j and c_2f_3 become

$$\bar{c}_1 = a_2x_1, \quad \bar{c}_2 = -a_1x_4 + a_2x_3 + a_4x_1, \quad \bar{c}_3 = (-a_1x_3 + a_3x_1)x_1,$$
$$\bar{c}_4 = 2a_1x_3^2 + 3a_2x_1x_4 - 2a_3x_1x_3, \quad \bar{c}_5 = (3a_1x_3x_4 - 4a_2x_3^2 - 3a_3x_1x_4)x_1,$$
$$\bar{c}_6 = (-6a_1x_3^2x_4 + a_2(-9x_1x_4^2 + 4x_3^3) + 18a_3x_1x_3x_4 - 12a_4x_1x_3^2)x_1,$$
$$\bar{c}_7 = (-8a_1x_3^4 - 18a_2x_1x_3^2x_4 + 2a_3(-9x_1x_4^2 + 4x_3^3)x_1 + 18a_4x_1^2x_3x_4)x_1,$$
$$\bar{c}_2\bar{f}_3 = (-3a_1x_4 + 3a_2x_3 + 3a_4x_1)x_1^2x_4.$$

Direct calculations give that the coordinates $\bar{v}_i$ of $v = \sum_{j=1}^{7} \bar{c}_j\bar{u}_j + \bar{c}_2\bar{f}_3\bar{u}_8 = 0$ are

$$-x_4\bar{u}_2 - x_1x_3\bar{u}_3 + 2x_3^2\bar{u}_4 + 3x_1x_3x_4\bar{u}_5 - 6x_1x_3^2x_4\bar{u}_6 - 8x_1x_3^4\bar{u}_7 - 3x_1^2x_4^2\bar{u}_8 = 0,$$
$$x_1\bar{u}_1 + x_3\bar{u}_2 + 3x_1x_4\bar{u}_4 - 4x_1x_3^2\bar{u}_5 + (-9x_1x_4^2 + 4x_3^3)x_1\bar{u}_6 - 18x_1^2x_3^2x_4\bar{u}_7 + 3x_1^2x_3x_4\bar{u}_8 = 0,$$
$$(x_1\bar{u}_3 - 2x_3\bar{u}_4 - 3x_1x_4\bar{u}_5 + 18x_1x_3x_4\bar{u}_6 + 2(-9x_1x_4^2 + 4x_3^3)\bar{u}_7)x_1 = 0,$$
$$(\bar{u}_2 - 12x_1x_3^2\bar{u}_6 + 18x_1^2x_3x_4\bar{u}_7 + 3x_1^2x_4\bar{u}_8)x_1 = 0,$$

where $\bar{u}_j = u_j(\bar{f}_1, \bar{f}_2, \bar{f}_4)$ Since $x_1v_1 + x_2v_2 + x_3v_3 + x_4v_4 = 0$, because v belongs to the commutator ideal L'_4/L''_4, we have that $x_1\bar{v}_1 + x_3\bar{v}_3 + x_4\bar{v}_4 = 0$. Hence we can remove the first equation and obtain

$$(x_1\bar{u}_1 + x_3\bar{u}_2 - 4x_1x_3^2\bar{u}_5 + (-9x_1x_4^2 + 4x_3^3)x_1\bar{u}_6) + 3(\bar{u}_4 - 6x_1x_3^2\bar{u}_7 + x_1x_3\bar{u}_8)x_1x_4 = 0,$$

$$(x_1\bar{u}_3 - 2x_3\bar{u}_4 + 2(-9x_1x_4^2 + 4x_3^3)\bar{u}_7) + 3(-\bar{u}_5 + 6x_3\bar{u}_6)x_1x_4 = 0,$$

$$(\bar{u}_2 - 12x_1x_3^2\bar{u}_6) + 3(6x_3\bar{u}_7 + \bar{u}_8)x_1^2x_4 = 0.$$

The variable x_4 participates in the polynomials $\bar{f}_1, \bar{f}_2, \bar{f}_4$ in even degrees only. Hence $\bar{u}_1, \ldots, \bar{u}_8$ do not contain odd degrees of x_4. The only odd degrees of x_4 in the above equations come from $3(\bar{u}_4 - 6x_1x_3^2\bar{u}_7 + x_1x_3\bar{u}_8)x_1x_4$, $3(-\bar{u}_5 + 6x_3\bar{u}_6)x_1x_4$, and $3(6x_3\bar{u}_7 + \bar{u}_8)x_1^2x_4$. Hence

$$(-\bar{u}_5 + 6x_3\bar{u}_6)x_1 = -\bar{f}_1\bar{u}_5(\bar{f}_1, \bar{f}_2, \bar{f}_4) - 3\bar{f}_2\bar{u}_6(\bar{f}_2, \bar{f}_4) = 0,$$

$$(6x_3\bar{u}_7 + \bar{u}_8)x_1 = -3\bar{f}_2\bar{u}_7(\bar{f}_2, \bar{f}_4) + \bar{f}_1\bar{u}_8(\bar{f}_1, \bar{f}_2, \bar{f}_4) = 0.$$

Since $\bar{f}_1, \bar{f}_2, \bar{f}_4$ are algebraically independent in $K[x_1, x_3, x_4]$, the equations

$$-\bar{f}_1\bar{u}_5(\bar{f}_1, \bar{f}_2, \bar{f}_4) - 3\bar{f}_2\bar{u}_6(\bar{f}_2, \bar{f}_4) = -3\bar{f}_2\bar{u}_7(\bar{f}_2, \bar{f}_4) + \bar{f}_1\bar{u}_8(\bar{f}_1, \bar{f}_2, \bar{f}_4) = 0$$

give that $\bar{u}_6 = \bar{u}_7 = 0$ and, as a consequence, $\bar{u}_j = 0$ for $j = 1, \ldots, 8$. Using again the algebraic independence of $\bar{f}_1, \bar{f}_2, \bar{f}_4$ we obtain that $u_j = 0$ for $j = 1, \ldots, 8$. This completes the proof that the $K[X_4]^\delta$-module $(L'_4/L''_4)^\delta$ is generated by $c_1, \ldots, c_7$. As in the previous example we obtain that the algebra $(L_4/L''_4)^\delta$ is generated by

$$\{x_1, c_jf_2^{r_j}f_4^{s_j}, c_2f_2^{r_2}f_3f_4^{s_2} \mid r_j, s_j \geq 0, j = 1, \ldots, 7\}.$$

Nowicki [22] conjectured that if all Jordan cells of the Weitzeböck derivation δ are of size 2×2, i.e., $\delta = \delta(1, \ldots, 1)$, then $K[X_{2d}]^\delta$ is generated by

$$\{x_{2j-1}, x_{2k-1}x_{2l} - x_{2k}x_{2l-1} \mid j = 1, \ldots, d, 1 \leq k < l \leq d\}.$$

There are several proofs of the conjecture based on different ideas. The unpublished proof by Derksen and the proof by Bedratyuk [3] show that the result follows from well known results of classical invariant theory. Khoury [17, 18] uses Gröbner bases techniques. The proof by Drensky and Makar-Limanov [15] is based on elementary ideas and the approach by Kuroda [20] exploits earlier ideas of Kurano [19] related with the Roberts counterexample to the Hilbert 14th problem [23]. In particular, [15] gives the Gröbner basis of the ideal of relations between the generators of the algebra $K[X_{2d}]^\delta$ and a basis for $K[X_{2d}]^\delta$ as a vector space. The next examples handle the cases $(L_4/L''_4)^\delta$, $\delta = \delta(1, 1)$ and $(L_6/L''_6)^\delta$, $\delta = \delta(1, 1, 1)$.

Example 5.3. Let $d = 4$, $\delta = \delta(1, 1)$, and let $\delta(x_1) = 0$, $\delta(x_2) = x_1$, $\delta(x_3) = 0$, $\delta(x_4) = x_3$. Then, see [22] and the comments above, $K[X_4]^\delta$ is generated by the algebraically independent polynomials $f_1 = x_1$, $f_2 = x_3$, $f_3 = x_1x_4 - x_2x_3$. Hence

$$H_{GL_2}(K[X_4]^\delta, t_1, t_2, z) = \frac{1}{(1 - t_1z)^2(1 - t_1t_2z^2)}.$$

By Example 3.2

$$H_{GL_2}((L'_4/L''_4)^\delta, t_1, t_2, z) = \frac{t_1z^2(t_1 + 3t_2 - t_1^2t_2z^2)}{(1 - t_1z)^2(1 - t_1t_2z^2)}.$$

The Lie elements

$$c_1 = [x_3, x_1], \quad c_2 = [x_2, x_1], \quad c_3 = [x_4, x_3], \quad c_4 = [x_4, x_1] - [x_3, x_2]$$

belong to $(L'_4/L''_4)^\delta$ and are of bidegree $(2,0)$ for c_1 and $(1,1)$ for the other three elements. It is easy to see that c_1,c_2,c_3,c_4 satisfy the relation

$$c_1f_3+c_2f_2^2+c_3f_1^2-c_4f_1f_2=0.$$

The $K[X_4]^\delta$-module generated by c_1,c_2,c_3,c_4 is spanned by the products $c_jf_1^{q_j}f_2^{r_j}f_3^{s_j}$, $q_j,r_j,s_j\geq 0$. The above relation gives that we can express the elements $c_1f_1^{q_1}f_2^{r_1}f_3^{s_1}$ with $s_1>0$ by elements which do not contain the factor c_1f_3. Hence we may assume that $s_1=0$. The generating function of the set

$$C=\{c_1f_1^{q_1}f_2^{r_1},c_jf_1^{q_j}f_2^{r_j}f_3^{s_j}\mid q_1,r_1,q_j,r_j,s_j\geq 0,j=2,3,4\}$$

is

$$G(C,t_1,t_2,z)=\frac{t_1^2z^2}{(1-t_1z)^2}+\frac{3t_1t_2z^2}{(1-t_1z)^2(1-t_1t_2z^2)}=H_{GL_2}((L'_4/L''_4)^\delta,t_1,t_2,z).$$

Hence, if we show that the elements of the set C are linearly independent we shall conclude that the $K[X_4]^\delta$-module $(L'_4/L''_4)^\delta$ is generated by c_1,c_2,c_3,c_4. Let

$$c_1u_1(f_1,f_2)+c_2u_2(f_1,f_2,f_3)+c_3u_3(f_1,f_2,f_3)+c_4u_4(f_1,f_2,f_3)=0$$

for some $u_1(f_1,f_2),u_j(f_1,f_2,f_3)\in K[f_1,f_2,f_3]$, $j=2,3,4$. Working in the wreath product $A_4\mathrm{wr}B_4$ we obtain

$$\begin{aligned}0&=(a_3x_1-a_1x_3)u_1(x_1,x_3)+(a_2x_1-a_1x_2)u_2(x_1,x_3,x_1x_4-x_2x_3)\\&\quad+(a_4x_3-a_3x_4)u_3(x_1,x_3,x_1x_4-x_2x_3)\\&\quad+(a_4x_1-a_1x_4-a_3x_2+a_2x_3)u_4(x_1,x_3,x_1x_4-x_2x_3)\\&=a_1(-x_3u_1(x_1,x_3)-x_2u_2(x_1,x_3,x_1x_4-x_2x_3)-x_4u_4(x_1,x_3,x_1x_4-x_2x_3))\\&\quad+a_2(x_1u_2(x_1,x_3,x_1x_4-x_2x_3)+x_3u_4(x_1,x_3,x_1x_4-x_2x_3))\\&\quad+a_3(x_1u_1(x_1,x_3)-x_4u_3(x_1,x_3,x_1x_4-x_2x_3)-x_2u_4(x_1,x_3,x_1x_4-x_2x_3))\\&\quad+a_4(x_3u_3(x_1,x_3,x_1x_4-x_2x_3)+x_1u_4(x_1,x_3,x_1x_4-x_2x_3)).\end{aligned}$$

In the coefficient of a_1 (which has to be equal to 0), the only expression which does not depend on x_2 and x_4 is $-x_3u_1(x_1,x_3)$ and hence $u_1=0$. This implies that $-x_2u_2-x_4u_4=0$ and $u_2=x_4u$, $u_4=-x_2u$ for some $u\in K[X_4]$. Similarly, from the coefficient of a_2 we derive $u_2=x_3v$, $u_4=-x_1v$ for some $v\in K[X_4]$. It follows from the equalities

$$u_2=x_4u=x_3v,\quad u_4=-x_2u=-x_1v$$

that $u_2=u_4=0$ which also implies that $u_3=0$. In this way the $K[X_4]^\delta$-module $(L'_4/L''_4)^\delta$ is generated by c_1,c_2,c_3,c_4. This also gives that the algebra $(L_4/L''_4)^\delta$ is generated by

$$\{x_1,c_1f_2^{r_1},c_jf_2^{r_j}f_3^{s_j}\mid r_1,r_j,s_j\geq 0,j=2,3,4\}.$$

Example 5.4. Let $d=6$, $\delta=\delta(1,1,1)$, and let $\delta(x_1)=\delta(x_3)=\delta(x_5)=0$, $\delta(x_2)=x_1$, $\delta(x_4)=x_3$, $\delta(x_6)=x_5$. Then, see [22] and [15], $K[X_6]^\delta$ is generated by the polynomials

$$f_1=x_1,\quad f_2=x_3,\quad f_3=x_5,$$
$$f_4=x_1x_4-x_2x_3,\quad f_5=x_1x_6-x_2x_5,\quad f_6=x_3x_6-x_4x_5,$$

with the only defining relation

$$\begin{vmatrix}x_1&x_3&x_5\\x_1&x_3&x_5\\x_2&x_4&x_6\end{vmatrix}=f_1f_6-f_2f_5+f_3f_4=0.$$

Hence we can replace f_2f_5 with $f_1f_6+f_3f_4$ and $K[X_6]^\delta$ has a basis

$$\{f_1^{q_1}f_2^{q_2}f_3^{q_3}f_4^{q_4}f_6^{q_6}, f_1^{q_1}f_3^{q_3}f_4^{q_4}f_5^{q_5+1}f_6^{q_6}\mid q_j\geq 0\}.$$

The Hilbert series of $K[X_6]^\delta$ is

$$H_{GL_2}(K[X_6]^\delta,t_1,t_2,z)=\frac{1}{(1-t_1z)^3(1-t_1t_2z^2)^2}+\frac{t_1t_2z^2}{(1-t_1z)^2(1-t_1t_2)^3}.$$

By Example 3.2

$$H_{GL_2}((L_6'/L_6'')^\delta,t_1,t_2,z)=\frac{t_1z^2p(t_1,t_2,z)}{(1-t_1z)^3(1-t_1t_2z^2)^3},$$

$$p=3(t_1+2t_2)+t_1(-t_1+t_2)z-9t_1^2t_2z^2+3t_1^2t_2(-3t_2+t_1)z^3$$
$$+t_1^2t_2^2(9t_1-t_2)z^4+3t_1^3t_2^2(t_2-t_1)z^5-3t_1^4t_2^3z^6+t_1^5t_2^3z^7).$$

Following the approach in Example 5.2 we have found a set of ten generators of the $K[X_6]^\delta$-module $(L_6'/L_6'')^\delta$:

$$c_1=[x_3,x_1],\quad c_2=[x_5,x_1],\quad c_3=[x_5,x_3],$$
$$c_4=[x_2,x_1],\quad c_5=[x_4,x_3],\quad c_6=[x_6,x_5],$$
$$c_7=[x_4,x_1]-[x_3,x_2],\quad c_8=[x_6,x_1]-[x_5,x_2],\quad c_9=[x_6,x_3]-[x_5,x_4],$$
$$c_{10}=[x_3,x_2,x_5]-[x_5,x_2,x_3]-[x_4,x_1,x_5]+[x_5,x_1,x_4]$$

and 21 relations between them:

$$R_1(3,0):c_3f_1=-c_1f_3+c_2f_2,$$
$$R_2(3,1):c_1f_4=-c_4f_2^2-c_5f_1^2+c_7f_1f_2,$$
$$R_3(3,1):c_1f_5=-c_2f_4-2c_4f_2f_3+c_7f_1f_3+c_8f_1f_2-c_9f_1^2,$$
$$R_4(3,1):c_3f_4=-c_1f_6+2c_5f_1f_3-c_7f_2f_3+c_8f_2^2-c_9f_1f_2,$$
$$R_5(3,1):c_3f_5=-c_2f_6-2c_6f_1f_2-c_7f_3^2+c_8f_2f_3+c_9f_1f_3,$$
$$R_6(3,1):c_3f_6=-c_5f_3^2-c_6f_2^2+c_9f_2f_3,$$
$$R_7(3,1):c_2f_5=-c_4f_3^2-c_6f_1^2+c_8f_1f_3,$$
$$R_8(3,1):c_{10}f_1=-c_1f_5-c_4f_2f_3+c_8f_1f_2-c_9f_1^2,$$
$$R_9(3,1):c_{10}f_2=-c_1f_6+c_5f_1f_3-c_7f_2f_3+c_8f_2^2-c_9f_1f_2,$$
$$R_{10}(3,1):c_{10}f_3=-c_2f_6-c_6f_1f_2-c_7f_3^2+c_8f_2f_3,$$
$$R_{11}(4,1):c_1f_1f_6=-c_2f_2f_4-c_4f_2^2f_3+c_5f_1^2f_3+c_8f_1f_2^2-c_9f_1^2f_2,$$
$$R_{12}(4,1):c_1f_3f_6=c_2f_2f_6+c_5f_1f_3^2+c_6f_1f_2^2-c_9f_1f_2f_3,$$
$$R_{13}(4,1):c_2f_1f_6=-c_2f_3f_4-c_4f_2f_3^2-c_6f_1^2f_2+c_8f_1f_2f_3,$$
$$R_{14}(3,2):c_{10}f_4=c_4f_2f_6+c_5f_1f_5-c_7(f_1f_6+f_3f_4)+c_8f_2f_4-c_9f_1f_4,$$
$$R_{15}(3,2):c_{10}f_5=c_4f_3f_6-c_6f_1f_4-c_7f_3f_5+c_8f_3f_4,$$
$$R_{16}(3,2):c_{10}f_6=-c_5f_3f_5-c_6f_2f_4+c_9f_3f_4,$$
$$R_{17}(4,2):c_1f_6^2=c_5f_3(f_3f_4+2f_1f_6)+c_6f_2^2f_4-c_7f_2f_3f_6+c_8f_2^2f_6-c_9f_2(f_3f_4+f_1f_6),$$
$$R_{18}(4,2):c_2f_4^2=c_4f_2(-f_3f_4+f_1f_6)+c_5f_1^2f_5-c_7f_1^2f_6+c_8f_1f_2f_4-c_9f_1^2f_4,$$
$$R_{19}(4,2):c_2f_4f_6=-c_4f_2f_3f_6-c_5f_1f_3f_5-c_6f_1f_2f_4+c_7f_1f_3f_6+c_9f_1f_3f_4,$$
$$R_{20}(4,2):c_2f_6^2=c_5f_3^2f_5+c_6f_2(f_3f_4-f_1f_6)-c_7f_3^2f_6+c_8f_2f_3f_6-c_9f_3^2f_4,$$
$$R_{21}(3,3):c_4f_6^2=-c_5f_5^2-c_6f_4^2+c_7f_5f_6-c_8f_4f_6+c_9f_4f_5.$$

Hence the $K[X_6]^\delta$-module generated by $\{c_1,\ldots,c_{10}\}$ is spanned by

$$c_1f_1^{q_1}f_2^{q_2}f_3^{q_3},c_1f_2^{q_2}f_6;\quad c_2f_1^{q_1}f_2^{q_2}f_3^{q_3}f_4^{\varepsilon},\quad c_2f_2^{q_2}f_3^{q_3}f_6;$$

$$c_3f_2^{q_2}f_3^{q_3};\quad c_4f_1^{q_1}f_2^{q_2}f_3^{q_3}f_4^{q_4}f_6^{\varepsilon},\quad c_4f_1^{q_1}f_3^{q_3}f_4^{q_4}f_5^{q_5+1}f_6^{\varepsilon};$$
$$c_jf_1^{q_1}f_2^{q_2}f_3^{q_3}f_4^{q_4}f_6^{q_6},\quad c_jf_1^{q_1}f_3^{q_3}f_4^{q_4}f_5^{q_5+1}f_6^{q_6};\quad c_{10},$$

where $q_i\geq 0$, $i=1,\dots,6$, $j=5,6,7,8,9$, $\varepsilon=0,1$. The generating function of this set is equal to the Hilbert series $H_{GL_2}((L_6'/L_6'')^{\delta},t_1,t_2,z)$. Hence, as in the other examples in this section, it is sufficient to show that the set consists of linearly independent elements. Let

$$\sum_{j=1}^{10}c_ju_j=0,$$

where u_j are polynomials in $f_1,\dots,f_6$ of the form

$$\begin{gathered}
u_1=u_1'(f_1,f_2,f_3)+u_1''(f_2)f_6,\\
u_2=u_2'(f_1,f_2,f_3)+u_2''(f_1,f_2,f_3)f_4+u_2'''(f_2,f_3)f_6,\\
u_3=u_3(f_2,f_3),\\
u_4=u_4'(f_1,f_2,f_3,f_4)+u_4''(f_1,f_2,f_3,f_4)f_6\\
+u_4'''(f_1,f_3,f_4,f_5)f_5+u_4^{(iv)}(f_1,f_3,f_4,f_5)f_5f_6,\\
u_j=u_j'(f_1,f_2,f_3,f_4,f_6)+u_j''(f_1,f_3,f_4,f_5,f_6)f_5,\quad j=5,6,7,8,9,\\
u_{10}=\text{const}.
\end{gathered}$$

Clearly, we may assume that the linear dependence $\sum_{j=1}^{10}c_ju_j=0$ is homogeneous. Since there is no linear dependence of degree 3, we conclude that $u_{10}=0$. As in the previous examples, we shall work in the abelian wreath product $A_6\mathrm{wr}B_6$. As in Example 5.2 we shall denote by v_i the coordinate of a_i of $v\in A_6\mathrm{wr}B_6$. The six coordinates v_i of

$$v=\sum_{j=1}^{9}c_ju_j=\sum_{i=1}^{6}a_iv_i=0$$

define a linear homogeneous system

$$v_i=0,\quad i=1,\dots,6,$$

with unknowns $u_1,\dots,u_9$ and with a matrix

$$\left(\begin{array}{ccccccccc}
x_3 & -x_5 & 0 & -x_2 & 0 & 0 & -x_4 & -x_6 & 0\\
0 & 0 & 0 & x_1 & 0 & 0 & x_3 & x_5 & 0\\
x_1 & 0 & -x_5 & 0 & -x_4 & 0 & -x_2 & 0 & -x_6\\
0 & 0 & 0 & 0 & x_3 & 0 & x_1 & 0 & x_5\\
0 & x_1 & x_3 & 0 & 0 & -x_6 & 0 & -x_2 & -x_4\\
0 & 0 & 0 & 0 & 0 & x_5 & 0 & x_1 & x_3
\end{array}\right).$$

We solve the system by the Gauss method keeping the entries of the matrix in $K[X_6]$. Since $v\in L_6'/L_6''$ and $\sum_{i=1}^{6}x_iv_i=0$, we can remove the first row of the matrix. Then we bring the matrix in a triangular form

$$\left(\begin{array}{ccccccccc}
x_1 & 0 & -x_5 & 0 & -x_4 & 0 & -x_2 & 0 & -x_6\\
0 & x_1 & x_3 & 0 & 0 & -x_6 & 0 & -x_2 & -x_4\\
0 & 0 & 0 & x_1 & 0 & 0 & x_3 & x_5 & 0\\
0 & 0 & 0 & 0 & x_3 & 0 & x_1 & 0 & x_5\\
0 & 0 & 0 & 0 & 0 & x_5 & 0 & x_1 & x_3
\end{array}\right).$$

We multiply the first row by x_3 and add to it the fourth row multiplied by x_4. Similarly we multiply the second row by x_5 and add the fifth row multiplied by x_6:

$$\begin{pmatrix} x_1x_3 & 0 & -x_3x_5 & 0 & 0 & 0 & x_1x_4-x_2x_3 & 0 & -(x_3x_6-x_4x_5) \\ 0 & x_1x_5 & x_3x_5 & 0 & 0 & 0 & 0 & x_1x_6-x_2x_5 & x_3x_6-x_4x_5 \\ 0 & 0 & 0 & x_1 & 0 & 0 & x_3 & x_5 & 0 \\ 0 & 0 & 0 & 0 & x_3 & 0 & x_1 & 0 & x_5 \\ 0 & 0 & 0 & 0 & 0 & x_5 & 0 & x_1 & x_3 \end{pmatrix}.$$

The second row of the matrix gives the equation

$$x_1x_5u_2+x_3x_5u_3+(x_1x_6-x_2x_5)u_8+(x_3x_6-x_4x_5)u_9=0.$$

Since u_3 depends on x_3,x_5 only and the monomials of all other summands depend also on the other variables, we conclude that $u_3=0$. Let w_j be the component of u_j which does not depend on x_2,x_4,x_6, $j=7,8,9$. Since u_1,u_2 depend linearly on x_2,x_4,x_6, the first two rows of the matrix give the system

$$x_1x_3(u_1'+(x_3x_6-x_4x_5)u_1'')+(x_1x_4-x_2x_3)w_7-(x_3x_6-x_4x_5)w_9=0$$

$$x_1x_5(u_2'+(x_1x_4-x_2x_3)u_2''+(x_3x_6-x_4x_5)u_2''')+(x_1x_6-x_2x_5)w_8+(x_3x_6-x_4x_5)w_9=0.$$

Since $u_1',u_1'',u_2',u_2'',u_2''',w_7,w_8,w_9$ do not depend on x_2,x_4,x_6, we derive that $u_1'=u_2'=0$. We rewrite the system in the form

$$-x_3w_7x_2+x_5(-x_1x_3u_1''+w_9)x_4+x_3(x_1x_3u_1''-w_9)x_6=0$$

$$-x_5(x_1x_3u_2''+w_8)x_2+x_5(x_1^2u_2''-x_1x_5u_2'''-w_9)x_4+(x_1x_3x_5u_2'''+x_1w_8+x_3w_9)x_6=0$$

which implies

$$w_7=0,\quad w_8=-x_1x_3u_2'',\quad w_9=x_1x_3u_1'',\quad x_3u_1''-x_1u_2''+x_5u_2'''=0.$$

The latter equation gives that every monomial of u_2'' depends on x_1 or x_5 which is impossible because $u_1''=u_1''(x_3)$. Hence $u_1''=0$, $-x_1u_2''+x_5u_2'''=0$, and u_2''' depends on x_1 which is also impossible. Again, $u_2''=u_2'''=0$. Now the matrix of the system with unknowns $u_4,\dots,u_9$ becomes

$$\begin{pmatrix} x_1 & 0 & 0 & x_3 & x_5 & 0 \\ 0 & x_3 & 0 & x_1 & 0 & x_5 \\ 0 & 0 & x_5 & 0 & x_1 & x_3 \\ 0 & 0 & 0 & x_1x_4-x_2x_3 & 0 & -(x_3x_6-x_4x_5) \\ 0 & 0 & 0 & 0 & x_1x_6-x_2x_5 & x_3x_6-x_4x_5 \end{pmatrix}$$

and the solution of the system is

$$u_4=f_6^2v,\quad u_5=f_5^2v,\quad u_6=f_4^2v,$$

$$u_7=-f_5f_6v,\quad u_8=f_4f_6v,\quad u_9=-f_4f_5,$$

$v\in K[X_6]^\delta$. Hence f_6^2 divides

$$u_4=u_4'(f_1,f_2,f_3,f_4)+u_4''(f_1,f_2,f_3,f_4)f_6$$

$$+u_4'''(f_1,f_3,f_4,f_5)f_5+u_4^{(iv)}(f_1,f_3,f_4,f_5)f_5f_6$$

and therefore f_6 divides $u_4'+u_4'''f_5$. If we order the variables by $x_6>x_4>x_5>x_1>x_1>x_3$, then the leading monomial of $u_4'+u_4'''$ with respect to the lexicographical order is the leading monomial of $u_4'''(x_1,x_5,x_1x_4,x_1x_6)x_1x_6$ which cannot be divisible by f_6 with leading monomial x_3x_6. Hence $u_4'+u_4'''f_5=0$. Again, f_6 does not divide $u_4''+u_4^{(iv)}f_5$ and, as a result, f_6^2 cannot divide u_4. Hence $v=0$ and this completes the proof.

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References

[1] G. Almkvist, W. Dicks, E. Formanek, Hilbert series of fixed free algebras and noncommutative classical invariant theory, J. Algebra **93** (1985), 189-214.

[2] Yu. A. Bahturin, Identical Relations in Lie Algebras (Russian), Nauka, Moscow, 1985. Translation: VNU Science Press, Utrecht, 1987.

[3] L. Bedratyuk, A note about the Nowicki conjecture on Weitzenbock derivations, Serdica Math. J. **35** (2009), 311-316.

[4] A.Ya. Belov, Rationality of Hilbert series of relatively free algebras, Uspekhi Mat. Nauk **52** (1997), No. 2, 153-154. Translation: Russian Math. Surveys **52** (1997), 394-395.

[5] F. Benanti, S. Boumova, V. Drensky, G.K. Genov, P. Koev, Computing with rational symmetric functions and applications to invariant theory and PI-algebras, Serdica Math. J. **38** (2012), 137-188. http://www.math.bas.bg/serdica/2012/2012-137-188.pdf.

[6] A. Berele, Applications of Belov's theorem to the cocharacter sequence of p.i. algebras, J. Algebra **298** (2006), 208-214.

[7] A. Berele, Properties of hook Schur functions with applications to p. i. algebras, Adv. Appl. Math. **41** (2008), 52-75.

[8] C. De Concini, D. Eisenbud, C. Procesi, Young diagrams and determinantal varieties, Invent. Math. **56** (1980), 129-165.

[9] H. Derksen, G. Kemper, Computational Invariant Theory, Encyclopaedia of Mathematical Sciences, Invariant Theory and Algebraic Transformation Groups **130**, Springer-Verlag, Berlin, 2002.

[10] V. Drensky, Fixed algebras of residually nilpotent Lie algebras, Proc. Amer. Math. Soc. **120** (1994), 1021-1028.

[11] V. Drensky, Free Algebras and PI-Algebras, Springer-Verlag, Singapore, 1999.

[12] V. Drensky, Invariants of unipotent transformations acting on noetherian relatively free algebras, Serdica Math. J. **30** (2004), 395-404. http://www.math.bas.bg/serdica/2004/2004-395-404.pdf.

[13] V. Drensky, G.K. Genov, Multiplicities of Schur functions with applications to invariant theory and PI-algebras, C.R. Acad. Bulg. Sci. **57** (2004), No. 3, 5-10.

[14] V. Drensky, C.K. Gupta, Constants of Weitzenböck derivations and invariants of unipotent transformations acting on relatively free algebras, J. Algebra **292** (2005), 393-428.

[15] V. Drensky, L. Makar-Limanov, The conjecture of Nowicki on Weitzenböck derivations of polynomial algebras, J. Algebra Appl. **8** (2009), 41-51.

[16] E.B. Elliott, On linear homogeneous diophantine equations, Quart. J. Pure Appl. Math. **34** (1903), 348-377.

[17] J. Khoury, Locally Nilpotent Derivations and Their Rings of Constants, Ph.D. Thesis, Univ. Ottawa, 2004.

[18] J. Khoury, A Groebner basis approach to solve a conjecture of Nowicki, J. Symbolic Comput. **43** (2008), 908-922.

[19] K. Kurano, Positive characteristic finite generation of symbolic Rees algebras and Roberts' counterexamples to the fourteenth problem of Hilbert, Tokyo J. Math. **16** (1993), 473-496.

[20] S. Kuroda, A simple proof of Nowicki's conjecture on the kernel of an elementary derivation, Tokyo J. Math. **32** (2009), 247-251.

[21] P.A. MacMahon, Combinatory Analysis, vols. 1 and 2, Cambridge Univ. Press. 1915, 1916. Reprinted in one volume: Chelsea, New York, 1960.

[22] A. Nowicki, Polynomial Derivations and Their Rings of Constants, Uniwersytet Mikolaja Kopernika, Torun, 1994. www-users.mat.umk.pl/~anow/ps-dvi/pol-der.pdf.

[23] P. Roberts, An infinitely generated symbolic blow-up in a power series ring and a new counterexample to Hilbert's fourteenth problem, J. Algebra **132** (1990), 461-473.

[24] A.L. Shmel'kin, Wreath products of Lie algebras and their application in the theory of groups (Russian), Trudy Moskov. Mat. Obshch. **29** (1973), 247-260. Translation: Trans. Moscow Math. Soc. **29** (1973), 239-252.
[25] B. Sturmfels, Algorithms in Invariant Theory. 2nd ed. Texts and Monographs in Symbolic Computation, Springer-Verlag, Wien, 2008.
[26] R. Weitzenböck, Über die Invarianten von linearen Gruppen, Acta Math. **58** (1932), 231-293.
[27] G. Xin, A fast algorithm for MacMahon's partition analysis, Electron. J. Comb. **11** (2004), No. 1, Research paper R58.

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